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The algebra of the classical Hamiltonian mechanics as the closure of two finite-dimensional algebras

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Abstract. It is shown that any generator of the infinite-dimensional canonical transformation group in the Hamiltonian formulation of classical mechanics can be obtained as a linear combination of repeated commutators of generators of two finite-dimensional subgroups. This result has some structural similarity with the Ogievetsky theorem concerning the algebra of the general coordinate transformation group in the theory of relativity.

1. Introduction

In the structural analysis of physical systems, geometrisation and mappings are concepts of basic importance. The structure of observables and the properties of states of the considered system are mapped on the elements of some geometrical space and allow us to classify physical objects according to their transformation and invariance properties or according to the specific time evolution. The study of the interplay between geometrical structures and transformation properties for the elements of any physical system is a powerful tool for the analysis of existing theories and for the development of new ones.

The problem is that for the familiar physical theories the set of allowed continuous transformations is in general an infinite-dimensional one. This concerns, e.g., classical mechanics, quantum mechanics and relativistic field theories. It is therefore desirable to determine some finite-dimensional subgroups in the set of all admissible transformations with simple transformation properties of the physical objects in such a way that the result of any continuous transformation of these objects may be derived from the transformation behaviour under the subgroups only. Consider, for example, the theory of general relativity. Applying any element of the group of continuous transformations to the coordinates x_μ one has

$$x_\mu \rightarrow x'_\mu = f_\mu(x_\nu) \tag{1.1}$$

where $f_\mu(x_\nu)$ is some continuous-differentiable function.

The algebra of generators for the transformations (1.1) may be given by the operators (Ogievetsky 1973)

$$\mathrm{i} x_0^{n_0} x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_\nu \tag{1.2}$$

($n_\mu \geqslant 0$, $\mu = 0, 1, 2, 3$, $\partial_\nu = \partial/\partial x_\nu$). The Ogievetsky theorem for this algebra of generators for the group of general coordinate transformations (1.1) is as follows. Any generator (1.1) can be expressed as a linear combination of repeated commutators of generators of the special linear group SL(4, R) and of those of the conformal group C_{15}. The infinite-dimensional algebra of generators (1.2) is the closure of these two finite-dimensional Lie algebras (Ogievetsky 1973, Konopel'chenko 1975). The use of this theorem permits us to define statements for the construction of relativistic invariant theories (Borisov and Ogievetsky 1974, Borisov 1978). We consider in this paper the Hamiltonian formulation of classical mechanics based on the $2n$ canonical variables (q_i, p^j), $i, j = 1, \ldots, n$ (i.e. the considered systems have n degrees of freedom). The observables of this physical system are all functions $f(q_i, p^j) \in C^\infty$ and they form the (real) Heisenberg algebra $\mathscr{A}_{\rm H}$ (Landau and Lifshitz 1970). The diffeomorphism group of this physical system, i.e. the canonical transformation group of $\mathscr{A}_{\rm H}$, plays a basic role in the qualitative theory of Hamiltonian systems. The group elements characterise possible time evolutions, symmetry transformations, invariance conditions and so on for these closed physical systems.

Any one-parameter canonical transformation group of $\mathscr{A}_{\rm H}$ may be generated by a vector field L_f, which is determined by an element $f \in \mathscr{A}_{\rm H}$ using the Poisson bracket operation as follows (Jost 1974):

$$L_f = \{\cdot, f\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p^i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p^i} \right). \tag{1.3}$$

These vector fields L_f (1.3) are the elements of the algebra $\mathscr{A}_{\rm L}$ of inner derivations of the Heisenberg algebra $\mathscr{A}_{\rm H}$. Note that the vector fields (1.3) consist, in the general case, of $2n$ terms in contrast to the generators (1.2).

We prove in this paper that any generator (1.3) of canonical transformations in the Heisenberg algebra may be given as a linear combination of repeated commutators of generators from two Lie subalgebras of $\mathscr{A}_{\rm L}$—the Lie algebra of the symplectical group Sp($2n$, R) and a Lie algebra of some 'projective canonical group' of dimension $n(n+2)$. The algebra $\mathscr{A}_{\rm L}$ is the closure of these two Lie algebras.

This result has in its structure some similarity to the Ogievetsky theorem, but the physical systems, the geometrical structures correlated with them and the transformation groups are quite different in their physical and mathematical contents.

Whilst the structure of the Lie algebra of the Sp($2n$, R) is well known from literature (Moshinsky *et al* 1974, Rosensteel and Rowe 1976, 1977), the algebra of the 'projective canonical group' (we use the notation PCG) seems not have been used so far in the phase space.

The paper is organised as follows. In § 2 we briefly repeat some basic facts of the classical Hamilton approach to fix our notation. In § 3 the Lie algebras of the groups Sp($2n$, R) and PCG are characterised as subalgebras in the algebra of vector fields (1.3). The properties of the algebra sp($2n$, R) are collected from the literature and written in a suitable form for our further consideration. The PCG algebra is derived from the projective group in configuration space. We point out in § 5 some common properties and differences between this algebra and those of the group SU(n, 1) discussed as spectrum generating algebra of the n-dimensional harmonic oscillator (Hwa and Nuyts 1966, Cocho *et al* 1967). In § 4 we prove our theorem and draw some conclusions in § 5. The connection of our formulae with some usual expressions concerning canonical transformations is established in a complex basis of the phase space in appendices 1 and 2.

If not otherwise mentioned, we denote summation in the formulae by repeated indices. Lie groups are denoted by capital symbols, the Lie algebras by the corresponding small letters. If possible, we omit the arguments from functions defined on phase space, i.e. we write f instead of $f(q, p)$.

2. The Hamilton algebra of classical mechanics

We consider the classical mechanics of a closed physical system of pointlike massive particles moving according to Newtonian law. In the standard Hamiltonian formulation (Landau and Lifshitz 1970) the $2n$ dynamical variables $(q, p) = (q_1, \ldots, q_n, p^1, \ldots, p^n)$ with the Poisson brackets

$$\{q_i, q_j\} = 0 \qquad \{p^i, p^j\} = 0 \qquad \{q_i, p^j\} = -\{p^j, q_i\} = \delta_i^j \tag{2.1}$$

form a generating system for the elements of the Heisenberg algebra $\mathscr{A}_\mathrm{H}$. Any element $f = f(q, p)$ of this algebra may be generated from the basic observables (2.1) as a result of three operations: addition, multiplication with real numbers (we restrict our considerations to real observables) and an associative, commutative product (Grgin and Petersen 1970, 1974). Owing to the measuring process in mechanical systems, the observables $f(q, p)$ may be represented by number-valued functions defined on the $2n$ real coordinates $(q, p) = (q_1, \ldots, q_n, p^1, \ldots, p^n)$ of the phase space.

The Poisson bracket operation (2.1) generates a skew-symmetric binary product between any two observables defined on the phase space as

$$\{f, g\} = \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q_i}\right). \tag{2.2}$$

The Poisson bracket operation (2.2) satisfies the requirements for a Lie product including the Jacobi identity implying that $\mathscr{A}_\mathrm{H}$ has the algebraic structure of a Lie ring.

The evolution law of the basic observables (2.1) and for any other observables $f \in \mathscr{A}_\mathrm{H}$ is given in the canonical form as

$$\dot{p}^k = \{p^k, H\} \qquad \dot{q}_k = \{q_k, H\} \qquad \dot{f} = \{f, H\} \tag{2.3}$$

where $H \in \mathscr{A}_\mathrm{H}$ is the Hamiltonian characterising the special dynamical system.

Any continuous transformation of the variables (2.1)

$$Q = Q(q, p, \tau) \qquad P = P(q, p, \tau) \tag{2.4}$$

(τ is a real parameter) is called canonical if it preserves the form of the evolution equations (2.3), i.e. the Poisson bracket (2.2). The finite transformations (2.4) may be expanded for an infinitesimal parameter ε as (Testa 1970, 1973)

$$Q = q + \varepsilon\{q, k\} \qquad P = p + \varepsilon\{p, k\}. \tag{2.5a}$$

For any element g of $\mathscr{A}_\mathrm{H}$ the infinitesimal transformation follows as

$$g(Q, P) = g(q, p) + \varepsilon\{g, k\} = g + \varepsilon L_k(g). \tag{2.5b}$$

The function k is called the generating element of the transformation (2.4). The evolution law (2.3) is clearly a special case of (2.5). The vector fields L_f constitute the elements of the infinite-dimensional algebra $\mathscr{A}_\mathrm{L}$ of inner derivations of the Heisenberg algebra $\mathscr{A}_\mathrm{H}$ of observables.

Owing to the Jacobi identity for the Poisson brackets (2.2) in $\mathscr{A}_\mathrm{H}$ it follows that for the linear map $\phi : \mathscr{A}_\mathrm{H} \to \mathscr{A}_\mathrm{L}$, i.e. the relation

$$f \to L_f = \{\cdot, f\} \qquad L_f(g) = \{g, f\} \tag{2.6}$$

there holds $\{f, g\} \to [L_f, L_g]$. In fact, the homeomorphism (2.6) between the two algebras $\mathscr{A}_\mathrm{H}$ and $\mathscr{A}_\mathrm{L}$ is the algebraical formulation of the duality principle, valid in the Hamilton approach of classical mechanics. The algebraical structures on observables and on generators of canonical transformations are realised by homeomorphic operations—as Poisson bracket and commutator respectively—on the same vector space structure of elements $f=f(q,p)$. The algebra $\mathscr{A}_{\rm H}$ of observables and the algebra $\mathscr{A}_{\rm L}$ of inner derivations are two realisations of the Hamilton algebra (Grgin and Petersen 1970, 1974) as the structural basis of an algebraic-geometrical formulation in classical mechanics.

The kernel of the canonical projection (2.6) of $\mathscr{A}_{\rm H}$ onto $\mathscr{A}_{\rm L}$ consists of the constant elements (the numbers) of $\mathscr{A}_{\rm H}$. Using (2.2) and (2.6) we get the analytical form (1.3) of the vector field L_f determined by the generating function f.

We shall in the following take advantage of the duality principle and consider the Lie algebras in the basis of generating functions instead of the $2n$-component vector fields L_f (1.3) themselves, i.e. we use the terms 'generating function' and 'generator' in a synonymous manner. In this way we get a more transparent form for the equations and for the proof of our theorem. A basis set in the algebra $\mathscr{A}_{\rm H}$ consists of the monomials in (q,p). Any element $f(q,p)$ may be given as

$$f(q,p)=a^{i_1\dots i_n}_{k_1\dots k_n}q_1^{i_1}\dots q_n^{i_n}\cdot(p^1)^{k_1}\dots(p^n)^{k_n}=a^{(i)}_{(k)}Q^{(i)}P^{(k)} \tag{2.7}$$

using (i) and (k) as symbols for the corresponding multi-index sets.

From (2.6) the corresponding basis follows for the generators of canonical transformations as

$$L^{i_1\dots i_n}_{k_1\dots k_n}(f)=L_{Q^{(i)}P^{(k)}}(f)=\{f,q_1^{i_1}\dots q_n^{i_n}(p^1)^{k_1}\dots(p^n)^{k_n}\}. \tag{2.8}$$

Finite canonical transformations (2.4) generated by means of monomials (2.8) have a simple explicit expression (Testa 1970, 1973, Dragt and Finn 1976, Heskia and Sofronion 1971, Stern 1978).

The set of all monomials of degree l $(l=\Sigma_{\nu=1}^n i_\nu+k_\nu)$ constitute a vector space $\mathscr{A}^l_{\rm H}$ of dimension $(2n+l-1)!/(2n-1)!l!$.

Note that the monomial basis in (2.7) is, together with (2.2), -2 graded. For $f_{l_1}\in\mathscr{A}^{l_1}_{\rm H}$, $f_{l_2}\in\mathscr{A}^{l_2}_{\rm H}$ it follows that

$$\{f_{l_1},f_{l_2}\}\in\mathscr{A}^{l_1+l_2-2}. \tag{2.9}$$

3. Two finite-dimensional Lie algebras of canonical transformations

Consider the functions $f(q,p)$ which are at most quadratic in the variables. They have the form (we omit a constant term)

$$f(q,p)=f_i^{(1)}q_i+f_j^{(2)}p^j+f_{jk}^{(1)}q_jq_k+f_{jk}^{(2)}q_jp^k+f_{jk}^{(3)}p^jp^k \tag{3.1}$$

$(f_i^{(\nu)},f_{jk}^{(\nu)}$ are real constants).

The elements (3.1) form a Lie subalgebra (it follows immediately from (2.9)).

The functions with $f_i^{(\nu)}=0$ $(\nu=1,2)$ generate the real symplectic group Sp$(2n,R)$ of dimension $n(2n+1)$ (Rosensteel and Rowe 1976, 1977).

The Lie algebra of this non-compact simple group is generated by the monomial basis

$$E_{ij}=-q_iq_j \qquad \text{dimension}\binom{n+1}{2} \tag{3.2a}$$

$$E^{ij}=p^ip^j \qquad \text{dimension}\binom{n+1}{2} \tag{3.2b}$$

$$E_i^j=q_ip^j \qquad \text{dimension } n^2. \tag{3.2c}$$

From (2.2) it follows that the generating functions (3.2a) and (3.2b) form two Abelian subalgebras separately. The generating functions (3.2c) satisfy the commutation relations

$$\{E_i^j, E_k^l\} = \delta_i^l E_k^j - \delta_k^j E_i^l \tag{3.3}$$

i.e. the algebra is isomorphic to the Lie algebra u(n) (see appendix 2). The set of generating functions (3.2c) constitutes the maximal compact subalgebra in the algebra sp($2n, R$).

Note that $E_0^0 = q_i p^i$ generates the u(1) subalgebra of u(n). This generator corresponds in the complex phase space basis (appendix 1) to the Hamiltonian of the harmonic oscillator. The remaining set of generators (3.2c) forms the (n^2-1)-dimensional Lie algebra su(n). A complete list of commutation relations for the Lie algebra (3.2) is given in appendix 2.

The elements of the symplectic group act as matrix transformations in any vector space $\mathscr{A}_H^l \subset \mathscr{A}_H$ (cf (2.9)). The translations t_{2n} are generated in $\mathscr{A}_H$ by the $2n$ variables (q_i, p^j) themselves. The Lie algebra (3.1) is therefore the semidirect sum sp($2n, R$) $\oplus$ t_{2n} generating the inhomogeneous group of canonical transformations in $\mathscr{A}_H$.

A further finite-dimensional canonical transformation group may be constructed starting with the broken linear transformations in the configuration space Q_n. These maps may be given as (Giovanni and Gliozzi 1965)

$$q_i' = \frac{a_i^l q_l + b_i}{a^l q_l + d} \tag{3.4}$$

i.e. they constitute the group of projective transformations in Q_n.

The real functions $f(q)$ defined on Q_n form a commutative subalgebra $\mathscr{A}_0$ in $\mathscr{A}_H$ (see (2.2)). The transformations (3.4) map the algebra $\mathscr{A}_0$ into itself.

(n^2+2n) independent vector fields corresponding to one-parameter subgroups in (3.4) may be given as follows:

$$\frac{\partial}{\partial q_i} \qquad n \text{ translation generators} \tag{3.5a}$$

$$-q_k\left(q_i\frac{\partial}{\partial q_i}\right) \qquad n \text{ generators of special conformal transformations} \tag{3.5b}$$

$$q_i\frac{\partial}{\partial q_j} \qquad n^2 \text{ generators.} \tag{3.5c}$$

The connection (2.5) and (2.6) of generating monomials and vector fields gives the possibility of expressing the generators (3.5) by generating monomials. Using $L_{p^k} = \partial/\partial q_k$, we obtain the generating monomials for (3.5) as

$$p^i \tag{3.6a}$$

$$-q_k(q_i p^i) = -q_k E_0^0 \tag{3.6b}$$

$$q_i p^j. \tag{3.6c}$$

The monomials (3.6), together with the Poisson bracket (2.2), form a Lie algebra which is isomorphic to the Lie algebra of the vector fields (3.5).

Using (2.6) the monomials (3.6) generate vector fields defined on all elements $f(q, p)$ of $\mathscr{A}_H$.

By writing for the monomials (3.6) the symbols

$$q_ip^j = A_i{}^j \qquad p^i = A_{n+1}{}^i \qquad q_iE_0^0 = A_i{}^{n+1} \qquad E_0^0 = A_{n+1}{}^{n+1} \tag{3.7}$$

one obtains the following commutation relation:

$$\{A_i{}^j, A_l{}^m\} = g_i{}^m A_l{}^j - g_l{}^j A_i{}^m \tag{3.8}$$

$$i, k, l, m = 1, \ldots, n+1 \qquad g_k^l = \delta_k^l \; (l, k = 1, \ldots, n) \qquad g_{n+1}^{n+1} = -1.$$

The generating monomials (3.6) therefore form a Lie algebra which is isomorphic to the algebra of the Lie group SU(n, 1). Comparing (3.6) and (3.7) in a complex basis (z, z^*) of the phase space (appendix 2) with the well known realisations of the Lie algebra su(n, 1) as a spectrum generating algebra of the harmonic oscillator (Hwa and Nuyts 1966, Cocho *et al* 1967), one finds as the essential difference that the generating monomials A_{n+1}^i are of first degree and the A_i^{n+1} of third degree respectively. The corresponding generators are not Hermitian conjugated (appendix 2).

Starting with the projective group (3.4) in configuration space, we derived a finite-dimensional group of canonical transformations defined on the whole algebra $\mathscr{A}_{\mathrm{H}}$. We therefore call the Lie group generated by the monomials (3.6) the projective canonical group (PCG) of the Heisenberg algebra $\mathscr{A}_{\mathrm{H}}$.

4. Proof of the theorem concerning the structure of the algebra $\mathscr{A}_{\mathrm{L}}$

We prove the following theorem. Any generator of the algebra $\mathscr{A}_{\mathrm{L}}$ of canonical transformations may be expressed as linear combination of repeated commutators of generators of the two subgroups Sp($2n$, R) and PCG. The algebra $\mathscr{A}_{\mathrm{L}}$ is the closure of these two Lie algebras of canonical transformations in $\mathscr{A}_{\mathrm{H}}$.

Corollary. Any element $f(q, p) \in \mathscr{A}_{\mathrm{H}}$ may be generated from the monomials (3.2) and (3.7) using the Poisson bracket (2.2). Consider the generating monomials of the Lie algebras sp($2n$, R) and PCG, i.e. the monomials (3.2) and (3.7) respectively

$$\begin{aligned} &q_ip^j, -q_iq_j, p^ip^j \qquad &&\text{dimension } 2n^2+n \\ &q_ip^j, p^i, q_iE_0^0 \qquad &&\text{dimension } n^2+2n. \end{aligned} \tag{4.1}$$

We prove the theorem by induction to the degree l of the generating monomials (2.8). The proof proceeds similar to that of the Ogievetsky theorem for generators (1.2) (Ogievetsky 1973, Konopel'chenko 1975). Referring to the closure of the two Lie algebras (4.1) as g, we show that g contains any monomial in q, p of third degree. We assume that g contains the monomials of some degree l (where l is a fixed integer, $l > 3$). It then follows that g also contains all monomials of degree $l+1$. Consequently g contains any monomial (2.8), i.e. g contains the same elements as $\mathscr{A}_{\mathrm{L}}$.

Firstly, g contains the elements $\{q_iq_j, p^i\} = q_j$, i.e. the inhomogeneous symplectic algebra (3.2) together with (4.1). Consider the Poisson brackets

$$\{q_iq_j, q_kE_0^0\} = 2q_iq_jq_k. \tag{4.2}$$

Taking all combinations of indices i, j, k we conclude that g contains all monomials in the variables q of third degree.

The generating monomials of the algebra contain all Poisson brackets

$$\{q_k E_0^0, p^i p^j\} = (p^j \delta_k^i + p^i \delta_k^j) E_0^0 + 2 p^i p^j q_k \tag{4.3}$$

i.e. the elements $p^i E_0^0$ and $p^i p^j q_k$ $(1 \leqslant i, j, k \leqslant n)$. From the Poisson brackets

$$\{q_i q_j, p^k E_0^0\} = (q_j \delta_i^k + q_i \delta_j^k) E_0^0 + 2 q_i q_j p^k \tag{4.4}$$

$(1 \leqslant i, l, k)$ we conclude that the generating functions $q_i q_j p^k$ belong to those of g too.

Using (4.3) we conclude that the generating functions of g contain $(1 \leqslant i, j, l \leqslant n)$

$$\{q_k p^i p^j, p^k p^l\} = p^i p^j p^l \tag{4.5}$$

i.e. all elements of third degree in p. Summarising (4.2)-(4.5) we conclude that the generating functions of the algebra g contain all elements of third degree in (q, p).

Assume then that the generating functions (2.8) of g contain all monomials of degree l in the variables (q, p), i.e. the elements of $\mathscr{A}_{\mathrm{H}}^l$

$$(Q)^{l_1}(P)^{l_2} \qquad l_1 + l_2 = l \qquad l_1 = 0, 1, \ldots, l. \tag{4.6}$$

Taking $l_1 = 0, 1, \ldots, l$ there follow $(l+1)$ different types of monomial (4.6) for each degree l. For the further proof we use (2.9).

The generating functions (2.8) contain, together with the elements (2.6) of the type $(Q)^l$, also the brackets

$$\{(Q)^l, q_j E_0^0\} = q_j \left(\sum_m q_m \frac{\partial}{\partial q_m} (Q)^l \right) \subset (Q)^{l+1}. \tag{4.7}$$

Taking all index combinations in $(Q)^l$ it follows that the generating functions also contain all monomials of the type $(Q)^{l+1}$.

Using (4.7) it follows that the monomial

$$\{q_i^{l+1}, (p^i)^2\} = 2(l+1) q_i^l p^i \tag{4.8}$$

is contained in the generating functions of g. Making use of the repeated Poisson brackets $\{q_i^l p^j, q_i p^k\}$ it follows that all monomials of the type $(Q)^l P$ are elements in the generating functions of g. The algebra g contains all generating functions of the form

$$\{(Q)^l p^i, p^j p^k\} \subset (Q)^{l-1}(P)^2 \tag{4.9}$$

and consequently all monomials of the type $(Q)^{l-1}(P)^2$ of degree $(l+1)$. Applying the generating functions $p^i p^j$ on the monomials of the type $(Q)^{l_1+1}(P)^{l_2}$ one gets the result that all monomials $(Q)^{l_1}(P)^{l_2+1}$ are also contained in g.

This procedure ends with the conclusion that all monomials of the type $(P)^{l+1}$ are contained in the set of generating functions of g. We conclude that, by proof of induction, the algebra g agrees with $\mathscr{A}_{\mathrm{L}}$. This proves our theorem. We summarise our results and give some comments.

5. Conclusions and remarks

We have shown that any generating function, i.e. any generator (1.3) or (2.8), can be expressed as a linear combination of repeated commutators of generators (4.1) of the symplectic group $\mathrm{Sp}(2n, R)$ and those of the projective canonical group PCG. The algebra $\mathscr{A}_{\mathrm{L}}$ is the closure of the two Lie algebras of these canonical transformation groups in the Heisenberg algebra $\mathscr{A}_{\mathrm{H}}$. The structural relations between the four algebras discussed above may be written in the following form:

$$\mathscr{A}_{\mathrm{L}} = \mathrm{sp}(2n, R) \cup \mathrm{PCG} \qquad \mathrm{u}(n) = \mathrm{sp}(2n, R) \cap \mathrm{PCG}. \tag{5.1}$$

Additional to the common subalgebra $\mathrm{u}(n)$ there are in both Lie algebras $\mathrm{sp}(2n, R)$ and PCG two further Abelian subalgebras. The generators of these subalgebras are transformed by the $\mathrm{u}(n)$ group as vectors or symmetrical tensors of second degree for the PCG or the $\mathrm{sp}(2n, R)$ algebra, respectively.

In proving the theorem we made use of the fact that the generating functions (3.2) of the symplectic group act in an irreducible manner on the elements of vector spaces $\mathscr{A}^{l}_{\mathrm{H}}$ (see (2.9)) whilst the generators $q_i E^0_0$ (3.6*b*) act as rising operators on the degree l, i.e. they map $(Q)^l$ into $(Q)^{l+1}$, and then from the action of $\mathrm{sp}(2n, R)$ follow all $Q^{l_1} P^{l_2}$.

Note that the PCG is a finite-dimensional subgroup in the conformal group, i.e. in the set of transformations $q_i = f_i(q)$, the $f(q)$ are analytical functions. The monomials $q_i p^j$, q_i and $p^i E^0_0$ $(i, j = 1, \ldots, n)$ generate a Lie algebra acting on the elements $f(p) \in \mathscr{A}_{\mathrm{H}}$ in a similar way as the PCG algebra acts on $f(q) \in \mathscr{A}_0$. This subalgebra of $\mathscr{A}_{\mathrm{L}}$ may be used instead of (4.1*b*) without changing the content of the theorem.

The Poisson brackets between $q_i E^0_0$ and $p^i E^0_0$ contain elements of fourth degree. Therefore they are not simultaneously elements of the same finite-dimensional Lie algebra.

Although the PCG algebra is isomorphic to those of the group $\mathrm{SU}(n, 1)$ ((A2.2) and (A2.3)) there is a remarkable difference between them, namely the realisations of the generators (3.6*a*) and (3.6*b*) or E^{n+1}_i and E^i_{n+1} respectively. It may therefore be useful to consider the PCG in connection with the dynamics of the harmonic oscillator too.

The structure scheme (5.1) may be compared with the Lie algebras concerning the Ogievetsky theorem, namely the algebra of generators for covariance transformations

$$\begin{aligned} \mathrm{so}(3,1) &= \mathrm{sl}(4, R) \cup \mathrm{so}(4,2) \\ &= \mathrm{sl}(4, R) \cap \mathrm{so}(4,2). \end{aligned} \tag{5.2}$$

The actions of the transformation groups occupying the same places in (5.1) and (5.2) have some common features considered in their geometrical structures. The Ogievetsky theorem concerns the diffeomorphism group of the space manifold with its pseudo-Riemannian geometry, whereas the theorem considered in our paper deals with the diffeomorphism group of Hamiltonian structures in classical mechanics, i.e. a symplectic geometry. The different basical bilinear forms in both geometries give rise to quite different geometrical properties in the physical systems and consequently to groups which are completely different from each other in both theorems.

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Appendix 1

Canonical transformations, their generating functions and the commutation relations between generators of subalgebras are often considered in a complex phase space basis (Moshinsky *et al* 1974, Hwa and Nuyts 1966, Cocho *et al* 1967). In order to transfer our results into this well known language, we consider the complex basic variables

$$z_i = \frac{1}{\sqrt{2}}(q_i + \mathrm{i}p^i) \qquad z_i^* = \frac{1}{\sqrt{2}}(q_i - \mathrm{i}p^i). \tag{A1.1}$$

The transformation (A1.1) maps the $2n$-dimensional real phase space into the complex space C_n. Any real function $f(q, p)$ is mapped onto a function defined on C_n. The observables of $\mathscr{A}_{\mathrm{H}}$ fulfil the reality condition as

$$f(z, z^*) = (f(z, z^*))^* = f^*(z^*, z). \tag{A1.2}$$

The Poisson bracket operation (2.1) and (2.2) is given as follows:

$$\{z_i, z_j^*\} = -\mathrm{i}\delta_{ij} \qquad \{f, g\} = -\mathrm{i}\sum_{k=1}^{n}\left(\frac{\partial f}{\partial z_k}\frac{\partial g}{\partial z_k^*} - \frac{\partial f}{\partial z_k^*}\frac{\partial g}{\partial z_k}\right). \tag{A1.3}$$

For any function $f(z, z^*)$ satisfying (A1.2) an infinitesimal canonical transformation (2.5) can be written as

$$\frac{\mathrm{d}z_k}{\mathrm{d}\varepsilon} = -\mathrm{i}\frac{\partial f}{\partial z_k} \qquad \frac{\mathrm{d}z_k^*}{\mathrm{d}\varepsilon} = \mathrm{i}\frac{\partial f}{\partial z_k^*}. \tag{A1.4}$$

Using (A1.3), any function $f(z, z^*)$ defines a complex vector field

$$L_f(g) = \{g, f\}. \tag{A1.5}$$

The generating monomials (3.2) of the algebra sp($2n$, R) and the corresponding vector fields may be given as

$$\begin{aligned} L_{z_i z_j} &= -\mathrm{i}\left(z_j\frac{\partial}{\partial z_i^*} + z_i\frac{\partial}{\partial z_j^*}\right) \\ L_{z_i^* z_j^*} &= -\mathrm{i}\left(z_i^*\frac{\partial}{\partial z_j} + z_j^*\frac{\partial}{\partial z_i}\right) \\ L_{z_i z_j^*} &= -\mathrm{i}\left(z_i\frac{\partial}{\partial z_j} - z_j^*\frac{\partial}{\partial z_i^*}\right). \end{aligned} \tag{A1.6}$$

For completeness we note the vector fields $L_{z_k^*} = -\mathrm{i}\,\partial/\partial z_k$ and $L_{z_k} = \mathrm{i}\,\partial/\partial z_k^*$.

The generating monomials (3.6*b*) of special conformal transformations give rise to vector fields with higher-order coefficient functions. The vector field $L_{z_l E_0^0}$ has the form

$$L_{z_l E_0^0} = -\mathrm{i}\sum_k z_l z_k\frac{\partial}{\partial z_k} + \mathrm{i}\sum_k(\delta_{lk}E_0^0 + z_l z_k^*)\frac{\partial}{\partial z_k^*}. \tag{A1.7}$$

Appendix 2

Using (A1.3), the Poisson brackets between the generating functions in each of the Lie algebras (3.2) and (3.6) may be evaluated directly. The result is for (3.2) (we use the notation $z_iz_j = E_{ij}$, $z_iz_j^* = E^j_i$, $z_i^*z_j^* = E^{ij}$)

$$\{E^j_i, E^l_k\} = \mathrm{i}(\delta^j_k E^l_i - \delta^l_i E^j_k) \tag{A2.1a}$$

$$\{E^0_0, E^{kl}\} = -2\mathrm{i}E^{kl} \qquad \{E^0_0, E_{kl}\} = 2\mathrm{i}E_{kl} \qquad \{E^0_0, E^j_i\} = 0$$

$$\begin{aligned}&\{E^{kl}, E_{mn}\} = \mathrm{i}(\delta^k_n E^l_m + \delta^l_n E^k_m + \delta^l_m E^k_n + \delta^k_m E^l_n)\\ &\{E^j_i, E_{kl}\} = \mathrm{i}(\delta^j_l E_{ik} + \delta^j_k E_{il})\\ &\{E^j_i, E^{kl}\} = -\mathrm{i}(\delta^k_i E^{jl} + \delta^l_i E^{jk}).\end{aligned} \tag{A2.1b}$$

For the generating monomials corresponding to (3.6), one has, beside (A2.1a), the Poisson brackets

$$\begin{aligned}&\{z_k^*, z_lE^0_0\} = \mathrm{i}(\delta_{kl}E^0_0 + z_lz_k^*)\\ &\{z_l^*, E^j_i\} = \mathrm{i}\delta_{il}z_j^* \qquad \{z_l^*, E^0_0\} = \mathrm{i}z_l^*\\ &\{z_kE^0_0, E^j_i\} = -\mathrm{i}\delta^j_k z_iE^0_0 \qquad \{z_kE^0_0, E^0_0\} = -\mathrm{i}z_kE^0_0.\end{aligned} \tag{A2.2}$$

Using the notation (see (3.7) and (3.8))

$$z_iz_j^* = A^j_i \qquad z_i^* = A^i_{n+1} \qquad z_iE^0_0 = A^{n+1}_i \qquad E^0_0 = A^{n+1}_{n+1} \tag{A2.3}$$

equations (A2.2) may be written in a compact form as follows:

$$\{A^j_i, A^m_l\} = \mathrm{i}(g^j_l A^m_i - g^m_i A^j_l) \tag{A2.4}$$

with $g^i_l = \delta^i_l$, $g^{n+1}_{n+1} = -1$ (see (3.7) and (3.8)). The generating monomials (3.6) of the PCG transformations close to a Lie algebra, which is isomorphic to the algebra of the group SU(n, 1).

Equations (3.7), (3.8) and (A2.3) are not the usual realisation of the Lie algebra su(n, 1), known in the context of dynamical algebras for the harmonic oscillator (Hwa and Nuyts 1966, Cocho *et al* 1967). The generators of translations and of special conformal transformations are not represented by Hermitian-conjugated operators. In the well known approaches (Hwa and Nuyts 1966, Cocho *et al* 1967) the generating functions take the form $z_i\,(E^0_0)^{1/2}$ and $z_i^*(E^0_0)^{1/2}$, respectively.

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